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The multiplicity of the two smallest distances among points

György Csizmadia*,[1]

Courant Institute, New York University, New York, NY 10012, USA

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Abstract

Let $1=d_1<d_2<\cdots<d_k$ denote the distinct distances determined by a set of n points in the plane. The multiplicity of the two smallest distances is smaller than $6n$ and it is maximized by the triangular lattice, where $d_2=\sqrt{3}$. We partially answer a question of Erdős and Vesztergombi by proving that $d_2\neq\sqrt{3}$ implies that the multiplicity of the two smallest distances is at most $4n$ unless d_2 is $(\sqrt{5}+1)/2$ or $1/(2\sin 15)$. In the case $d_2=(\sqrt{5}+1)/2$, the multiplicity is at most $4.5n$. We also show some extremal configurations for different values of d_2. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

In 1946, Paul Erdős posed the following — still unsolved — problem: determine the maximum number of unit distances in a set of n points in the plane [3]. Here we consider only configurations, where the minimal distance between two points is one.

In 1974, Harborth [5] proved that the multiplicity of the smallest distance among n points in the plane is at most $\lfloor 3n-\sqrt{12n-3}\rfloor$. Brass ([1,2], see also [10]) proved that the multiplicity of the second smallest distance is at most $\frac{24}{7}n$. Both of the above results are sharp.

Definition. Let X be a set of n points in the plane. Let $1=d_1<d_2<\cdots<d_k$ $(d_i=d_i(X))$ denote the distinct distances occurring between the points of X. Let m_i $(=m_i(X))$ denote the multiplicity of d_i $(\sum_{i=1}^{k} m_i=\binom{n}{2})$.

In this paper, I answer a question of Erdős and Vesztergombi by giving tight bounds on the maximum *combined* multiplicity of the two smallest distances among n points in the plane.

* E-mail: csiz@cims.nyu.edu.

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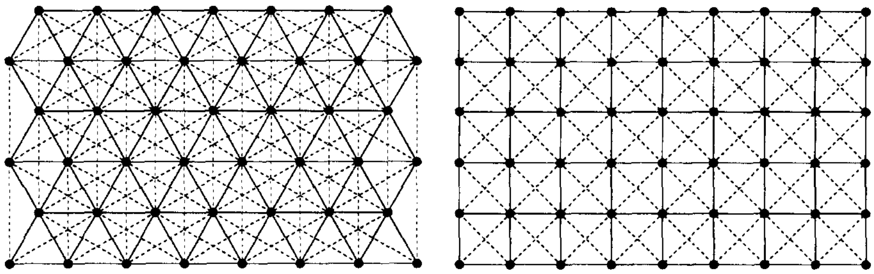

Fig. 1.

It is known that $m_1+m_2\leqslant 6n$ (see [10]) and the bound $6n$ is asymptotically attained by the triangular lattice, where $d_2=\sqrt{3}$ (see Fig. 1). In Theorems 1–4, I prove that m_1+m_2 is at most $4n$, unless the second smallest distance is equal to $\sqrt{3}$, $(\sqrt{5}+1)/2$ or $1/(2\sin 15)$. The bound $4n$ is asymptotically tight, too, as is shown by the square lattice (Fig. 1), in which there are exactly 8 points at distance d_1 or d_2 from each point. In general, one can establish an upper bound u by showing that there are at most $2u$ points at distance d_1 or d_2 from an 'average point'.

Theorem 1. *The multiplicity of the two smallest distances among n points in the plane is at most* $4n$, *provided that the ratio of the two smallest distances* (d_2/d_1) *is less than* $\sqrt{3}$, *but not the golden ratio. More precisely, we have*

$$\sup_n \max_{|X'|=n} \frac{m_1+m_2}{n}=4,$$

where the maximum is taken over all sets X' *of n points with* $(\sqrt{5}+1)/2\neq d_2<\sqrt{3}$.

Theorem 2. *The multiplicity of the two smallest distances among n points in the plane is at most* $4.5n$ *provided that the ratio of the two smallest distances is the golden ratio. More precisely, we have*

$$4\leqslant \sup_n \max_{|X''|=n} \frac{m_1+m_2}{n}\leqslant 4.5,$$

where the maximum is taken over all sets X'' *of n points with* $d_2=(\sqrt{5}+1)/2$. *Furthermore, there is a configuration which implies that the lower bound is asymptotically tight.*

Theorem 3.

$$\frac{33}{7}\leqslant \sup_n \max_{|X'''|=n} \frac{m_1+m_2}{n},$$

where the maximum is taken over all sets X''' *of n points with* $d_2=1/(2\sin 15)$.

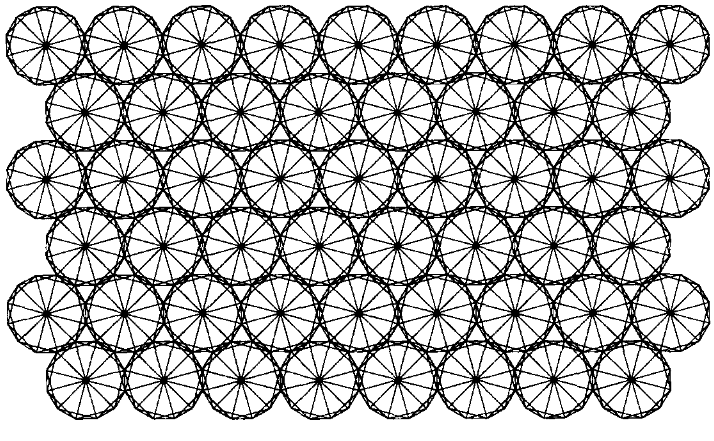

Fig. 2.

Theorem 4.

$$\sup_n \max_{|X''''|=n} \frac{m_1+m_2}{n} \leqslant 4,$$

where the maximum is taken over all sets X'''' *of n points with* $1/(2\sin 15) \neq d_2 > \sqrt{3}$.

The lower bound $\frac{33}{7}$ for the case $d_2 = 1/(2\sin 15)$ is shown by the same configuration for which $c_2 = \frac{24}{7}$ is attained (see Fig. 2).

The most exciting case is when $d_2 = (\sqrt{5}+1)/2$. Then I conjecture that the lower bound in Theorem 2 is tight. However, I have exhibited some configurations, in which a few points have degree 9 and the others degree 8. Although the average degree of the vertices does not exceed 8, the degrees of some points do, i.e., my conjecture is 'locally violated'.

2. Proof of Theorem 1

Proof of Theorem 1. The unit square lattice implies $4 \leqslant \sup_n \max_{|X'|=n}[(m_1+m_2)/n]$. Therefore we only have to prove that $\sup_n \max_{|X'|=n}[(m_1+m_2)/n] \leqslant 4$.

Let X be a set of n points in the plane where $d_1 = 1$, $d_2 < \sqrt{3}$ and $d_2 \neq (\sqrt{5}+1)/2$. Let $G = G(X)$ be the graph, whose vertices are the points of X, and two vertices are connected by an edge (drawn as a straight line segment), whenever their distance is either d_1 or d_2.

For any vertex $P \in G$, let $\deg_1(P)$ ($\deg_2(P)$) denote the number of neighbors of P, which are at distance d_1 (d_2, respectively) from P. Let $\deg(P) = \deg_1(P) + \deg_2(P)$.

We introduce the following notation.

Let ABC be a triangle where $|AB| = |AC| = d_2$ and $|BC| = 1$. Then let $\alpha = \alpha(d_2) := \angle CAB$ and $\delta = \delta(d_2) := \angle ABC = \angle BCA$ (see Fig. 3). Let DEF be a triangle

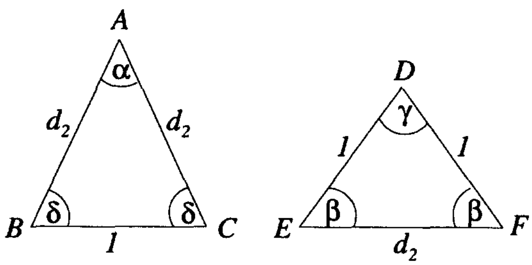

Fig. 3.

where $|DE| = |DF| = 1$ and $|EF| = d_2$. Then let $\beta = \beta(d_2) := \angle DEF = \angle EFD$ and $\gamma = \gamma(d_2) := \angle FDE$. In the sequel all angles are measured in degrees and we will drop the $^\circ$ notation.

The relations between these angles are

$$\beta = \arccos\left(\frac{1}{4\sin\frac{\alpha}{2}}\right), \qquad \gamma = 180 - 2\beta, \qquad \alpha = 180 - 2\delta.$$

By analyzing six cases, we will prove that the average degree of the vertices of G is at most 8. We will prove that there are no vertices of degree 10. For each vertex P of degree 9, we will assign some of its neighbors $X_1, \ldots, X_m$ of degree at most 7, so that if X_i is assigned to t_i vertices, then

$$\deg(P) + \sum_{i=1}^{m} \frac{\deg(X_i) - 8}{t_i} \leqslant 8$$

is satisfied. This implies that the average degree is indeed at most 8. We will call the left side of the inequality the *Adjusted Degree* of P.

Let P be a fixed vertex of G. Let $e_1, \ldots, e_k$ ($k \geqslant 9$, $e_{k+1} = e_1$) denote the edges adjacent to P in clockwise order and let $e_i = PX_i$. Let $\angle e_i e_j$ denote the clockwise angle from e_i to e_j. Let $|e_i|$ denote the length of e_i (i.e. $|e_i| = 1$ or $|e_i| = d_2$). If $|e_i| = 1$, then we will call e_i *short*, if $|e_i| = d_2$, then we will call it *long*.

Observe that if $\max(\alpha, \beta) \leqslant 36$, then $\beta = \min(\alpha, \beta) > 30$ and $\alpha \geqslant 33.5$ (since $d_2 < \sqrt{3}$). If $\max(\alpha, \beta) \geqslant 36$, then $\alpha = \min(\alpha, \beta) \geqslant 36$. Moreover $d_2 = (\sqrt{5} + 1)/2$ if and only if $\alpha = \beta = 36$. It is easy to see that $\angle e_i e_{i+1} \geqslant \min(\alpha, \beta)$. We will need the following rules.

Rule A. If $\alpha < 36$ and e_1, e_2 and e_3 are consecutive edges so that e_1 is short, then $\angle e_1 e_3 \geqslant 60 + \beta \geqslant 90$.

Proof. If e_3 is short, then either $\angle e_1 e_3 = 60$ or $\angle e_1 e_3 \geqslant \gamma > 60 + \beta$. But $\angle e_1 e_3 = \angle e_1 e_2 + \angle e_2 e_3 \geqslant 2\min(\alpha, \beta) > 60$, so the second case must hold.

Suppose that e_3 is long. Then $\angle e_1 e_3 > \beta$ implies that $\angle e_1 e_3 \geqslant \delta$. If $\angle e_1 e_2 \leqslant \max(\alpha, \beta)$ and $\angle e_2 e_3 \leqslant \max(\alpha, \beta)$, then $\angle e_1 e_3 \leqslant 2\max(\alpha, \beta) < \delta$, which is a contradic-

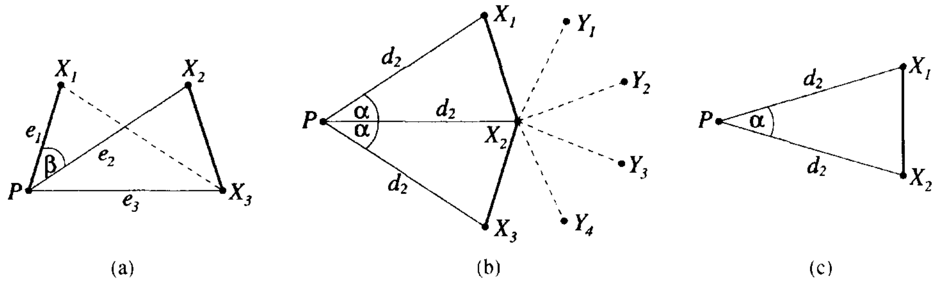

Fig. 4.

tion. Therefore, one of $\angle e_1e_2$ and $\angle e_2e_3$ should be greater than $\max(\alpha, \beta)$, thus at least 60. This implies the rule.

Rule B. If $\alpha > 36$ and e_1, e_2, e_3 are consecutive edges so that e_1 is short, e_2 and e_3 are long, then there is no edge connecting X_1 and X_3 (see Fig. 4B).

Proof. $\angle X_1PX_3 = \angle X_1PX_2 + \angle X_2PX_3 \geqslant \beta + \alpha > \delta$, therefore the segment X_1X_3 is longer than d_2.

Rule C. Suppose that vertex P has 3 consecutive neighbors X_1, X_2, and X_3, such that $|PX_1| = |PX_2| = |PX_3| = d_2$ and $\angle X_1PX_2 = \angle X_2PX_3 = \alpha$ (see Fig. 4C). Then

- if $\alpha < 36$, then X_2 has at most 6 neighbors;
- if $\alpha > 36$, then X_2 has at most 7 neighbors.

Proof. (i) If $\alpha < 36$, then it is easy to see that X_3, P and X_1 are consecutive neighbors of X_2. Suppose that X_2 has 4 other neighbors Y_1, Y_2, Y_3 and Y_4 (in clockwise order).

If one of the edges X_2Y_i $(1 \leqslant i \leqslant 4)$ is short (we may suppose that X_2Y_k), then $\angle X_3X_2P = \angle PX_2X_1 = \delta$, $\angle X_1X_2Y_k \geqslant \gamma$, $\angle Y_kX_2X_3 \geqslant \gamma$, so $360 \geqslant 2\delta + 2\gamma > 2 \cdot 72 + 2 \cdot 108 > 360$, a contradiction.

So we may suppose that X_2Y_i are all long edges. Then $\angle X_3X_2P = \angle PX_2X_1 = \delta$, $\angle Y_2X_2Y_3 \geqslant \alpha$ and by Rule A, $\angle X_1X_2Y_2 \geqslant 60 + \beta$ and $\angle Y_3X_2X_3 \geqslant 60 + \beta$, therefore,

$$360 \geqslant 2\delta + 2 \cdot (60 + \beta) + \alpha = 180 - \alpha + 120 + 2\beta + \alpha = 300 + 2\beta > 360,$$

which is a contradiction. So X_2 has at most 6 neighbors.

(ii) If $\alpha > 36$, then again X_3, P and X_1 are consecutive neighbors of X_2. Then

$$\frac{360 - \angle X_3X_2X_1}{6} = \frac{360 - 2\delta}{6} = \frac{180 + \alpha}{6} < \alpha$$

implies that X_2 has at most 4 neighbors different from X_3, P and X_1.

Rule D. If vertex P has 2 consecutive neighbors X_1 and X_2, such that $|PX_1| = |PX_2| = d_2$ and $\angle X_1PX_2 = \alpha$ (see Fig. 4D), then X_2 has at most 8 neighbors.

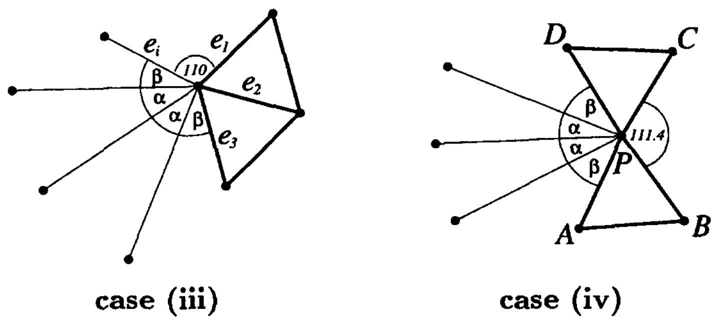

Fig. 5.

Proof. The proof is very similar to that of Rule C.

Now we return to the proof of Theorem 1. We distinguish the following six cases.

(1) $\deg(P) \geqslant 9$ *and* $\deg_1(P) \geqslant 4$.

Clearly $\angle e_i e_{i+1} \geqslant \min(\alpha, \beta)$ for all i, therefore $\alpha \leqslant \frac{360}{9} = 40$. This implies $\gamma > 94$. Four short edges are adjacent to P, and if e_i and e_j are short edges, then $\angle e_i e_j = 60$ or $\angle e_i e_j \geqslant \gamma$.

We distinguish four subcases:

(1.1) *No pair of short edges enclose angle* 60.

Then all pair of short edges enclose at least 94, but this is a contradiction, since $4 \cdot 94 > 360$.

(1.2) *Exactly one pair of short edges enclose angle* 60.

If e_i and e_j are short and $\angle e_i e_j = 60$, then $j = i + 1$. Therefore, we have 300° degrees left for the other edges, so $\frac{300}{8} = 37.5$ implies that $\alpha \leqslant 37.5$. Then $\gamma > 102$, and we know that any other pair of short edges enclose an angle at least γ. But $3 \cdot 102 + 60 > 360$ is a contradiction.

(1.3) *There are at least* 3 *consecutive short edges.*

If we had a fourth consecutive short edge, then the first and fourth would enclose at least $3 \cdot 60 = 180$ degrees, and there are at least 5 more edges left, which would imply $6 \cdot \min(\alpha, \beta) + 180 \leqslant 360$, which is impossible.

Let e_1, e_2, e_3 denote the three consecutive short edges, and let e_i denote a fourth one (see Fig. 5). Clearly we cannot have a fifth short edge e_j, because then $\angle e_i e_j \geqslant 60$ would imply $3 \cdot 60 + 2\gamma \leqslant 360$, which is impossible. Therefore, we have at least 5 long edges, 3 of which either immediately follow or immediately precede e_i. Without loss of generality, we may suppose that we have the latter scenario.

There are at least 6 edges following e_3 and preceeding e_1, so $min(\alpha, \beta) \leqslant \frac{360-120}{7} < 34.3$, which implies $\beta < 34.3$ and $\gamma > 111$. We also know that $\angle e_i e_1 \geqslant \gamma$, so we have $2\alpha + 2\beta + \gamma + 60 + 60 \leqslant 360$. This implies $\alpha + \beta \leqslant 64.5$, therefore $\beta = \min(\alpha, \beta) \leqslant 32.3$. Thus $\gamma \geqslant 115$, and we know that $\alpha > 33.5$, which contradicts $2\alpha + 2\beta + \gamma + 60 + 60 \leqslant 360$.

(1.4) *There are at least* 2 *disjoint pairs of short edges enclosing angle* 60.

Now $\min(\alpha,\beta)\leqslant\frac{360-120}{7}<34.3$, which implies $\beta<34.3$, $\alpha<36$, $\gamma>111.4$ (see Fig. 5).

We cannot have a fifth short edge, because it should enclose at least γ with both triangles PAB and PCD, but $3\cdot 111.4+120>360$. Therefore, we must have at least 5 long edges, at least 3 of which are consecutive. We may suppose that those 3 immediately follow PA in clockwise direction. Then $\angle CPB\geqslant\gamma$, so $2\alpha+2\beta+\gamma+60+60\leqslant 360$ must hold, but $2\beta+\gamma=180$ and $\alpha\geqslant 33.5$, which is a contradiction.

(2) $\deg(P)\geqslant 10$ *and* $\deg_1(P)\leqslant 3$.
It is easy to see that $\deg(P)\geqslant 10$ implies $\alpha\leqslant 36$. We distinguish two subcases.
(2.1) *There are no short edges* ($\deg_1(P)=0$).
Now if no consecutive pair of long edges enclose at least 60, then $\alpha=\frac{360}{10}=36$ or $\alpha\leqslant\frac{360}{11}<33.5$, both impossible. On the other hand, if two consecutive long edges enclose at least 60, then the others enclose at least α, but $\alpha\leqslant\frac{300}{9}=33.33<33.5$ is a contradiction.
(2.2) *There is at least one short edge* ($1\leqslant\deg_1(P)\leqslant 3$).
In this case let e_i be a short edge. Then $\angle e_{i-2}e_i\geqslant 60+\beta$ and $\angle e_ie_{i+2}\geqslant 60+\beta$ by Rule A. Thus $\beta\leqslant\frac{360-2(60+\beta)}{6}<30$, a contradiction.

(3) $\deg(P)=9$, $2\leqslant\deg_1(P)\leqslant 3$ *and* $\alpha<36$.
(3.1) *There are* 2 *consecutive short edges* (e_i *and* e_{i+1}).
Then $\angle e_{i-2}e_i\geqslant 60+\beta$ and $\angle e_{i+1}e_{i+3}\geqslant 60+\beta$, which implies $\beta\leqslant\frac{360-60-2(60+\beta)}{4}\leqslant 30$, a contradiction.
(3.2) *There are no* 2 *consecutive short edges.*
We may suppose that e_1 and e_k are short edges and $3\leqslant k\leqslant 6$. Now $\angle e_8e_1\geqslant 60+\beta$, $\angle e_1e_3\geqslant 60+\beta$ and $\angle e_ke_{k+2}\geqslant 60+\beta$ by Rule A. Moreover, $\angle e_3e_k+\angle e_{k+2}e_8\geqslant 3\beta$, which implies $360\geqslant 3\cdot(60+\beta)+3\beta=180+6\beta>360$, a contradiction.

(4) $\deg(P)=9,\deg_1(P)\leqslant 1$ *and* $\alpha<36$.
In this case there are at least 8 long edges. We may suppose that $|e_1|=|e_2|=\cdots=|e_8|=d_2$. We cannot have 3 pairs of consecutive edges enclosing at least 60, because then $\beta\leqslant\frac{360-180}{6}=30$ would follow, which is impossible.

Therefore we must have 2 edges $e_i(=PX)$ and $e_j(=PY)$ $(i\neq j)$, so that $|e_{i-1}|=|e_i|=|e_{i+1}|=d_2$, $|e_{j-1}|=|e_j|=|e_{j+1}|=d_2$ and $\angle e_{i-1}e_i=\angle e_ie_{i+1}=\angle e_{j-1}e_j=\angle e_je_{j+1}=\alpha$. Then, by Rule C, vertices X and Y have at most 6 neighbors each. However, two of their neighbors cannot have 9 neighbors, therefore both X and Y are assigned to at most 4 vertices. Thus, the Adjusted Degree of P is $9+2\cdot\frac{-2}{4}=8$, so we are done.

(5) $\deg(P)=9$ *and* $2\leqslant\deg_1(P)\leqslant 3$ *and* $\alpha>36$.
Now we know that $\beta>\alpha>36$, $\delta<72$ and $\gamma<108$. We distinguish two subcases.
(5.1) *There are two short edges* (e_i,e_{i+1}) *enclosing angle* 60.
Now $\alpha\leqslant\frac{300}{8}=37.5$. This implies $\delta>71.25$, $\beta<38.95$ and $\gamma>102$.

We cannot have any other consecutive pair of edges enclosing at least 60, becasue it would imply that $\alpha\leqslant\frac{240}{7}=34.4$, which is impossible. So for all $j\neq i$, $\angle e_je_{j+1}=\alpha$ or $\angle e_je_{j+1}=\beta$.

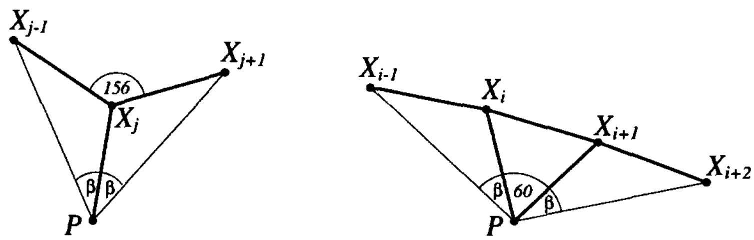

Fig. 6.

(5.1.a) If $\deg_1(P)=2$, then we have 7 consecutive long edges, 5 of which have the property that the edges immediately preceding and following them are both long, and enclose angle α with them. By Rule C these edges have degree at most 7, and at least 2 of its neighbors have degree at most 8. Thus the Adjusted Degree of P is $9+5\cdot\frac{-1}{5}=8$, so we are done.

(5.1.b) If $\deg_1(P)=3$, then let e_j be the third short edge. Since $\angle e_je_i>60$ and $\angle e_{i+1}e_j>60$, both of these angles are at least γ ($\gamma>102$). Now $\frac{360-60-\gamma}{6}<33$ and $\alpha>36$ implies that there are at least 2 long edges between these short ones, in both of these cases. Then $2\beta+\alpha>\gamma$ implies that $\angle e_{i+1}e_j>\gamma$ and $\angle e_je_i>\gamma$, so none of X_i or X_{i+1} is a neighbor of X_j.

By Rule B, X_j has now exactly 2 neighbors among the X_k ($1\leqslant k\leqslant 9$) and those are X_{j-1} and X_{j+1} (see Fig. 6).

But $\angle X_{j-1}X_jX_{j+1}=360-2\gamma\leqslant 156$ and $5\alpha>156$, which implies that X_j has at most 3 more neighbors. So X_j has altogehter at most 6 neighbors.

On the other hand, for the same reasons X_i has also only 2 neighbors among X_k ($1\leqslant k\leqslant 9$), and those are X_{i-1} and X_{i+1}. Now $\angle X_{i-1}X_iX_{i+1}=360-60-\gamma\leqslant 198$ and $6\alpha>198$, which implies that X_i has at most 4 more neighbors, so X_i has altogether at most 7 neighbors. The same is true for X_{i+1}.

Since we have at least 6 long edges, we must have at least two of them (e_k and e_l) so that e_{k-1},e_{k+1},e_{l-1} and e_{l+1} are all long edges. Then, by Rule C, it follows that X_k and X_l have at most 7 neighbors, and at most 5 of those neighbors can have degree 9.

Thus X_j contributes $\frac{-2}{6}$, X_i and X_{i+1} contribute $2\cdot\frac{-1}{7}$ and X_k and X_l contribute $2\cdot\frac{-1}{5}$ to the Adjusted Degree of P, which altogether gives $9-\frac{1}{3}-\frac{2}{7}-\frac{2}{5}<8$, so we are done.

(5.2) *There are no two short edges enclosing angle* 60.
Then all pairs of short edges enclose at least γ. Since $\frac{360-\gamma}{8}<36$, there are no two consecutive short edges.

If e_i and e_{i+2} were two short edges for some i, then $2\beta<\gamma\leqslant\angle e_ie_{i+2}=\angle e_ie_{i+1}+\angle e_{i+1}e_{i+2}$ would imply that either $\angle e_ie_{i+1}>\beta$ or $\angle e_{i+1}e_{i+2}>\beta$. We may suppose the first, which implies $\angle e_ie_{i+1}\geqslant\delta$. But then the inequality $\alpha\leqslant\frac{360-(\delta+\beta)}{7}$ would not be satisfied, which is a contradiction. Thus there are 2 long edges between any pair of short edges.

If e_i is short and e_{i+1} is long, then either $\angle e_ie_{i+1} = \beta$ or $\angle e_ie_{i+1} = \delta$. But $\frac{360-\delta}{8} < \alpha$, so $\angle e_ie_{i+1} = \beta$. Again by Rule B, (as in case (5.1.b)), if e_i is short, then X_i has at most 6 neighbors. So if there are 3 short edges, then we are done.

Suppose that there are only 2 short edges (e_i, e_j). We know that consecutive short and long edges enclose β. We cannot have 2 pairs of consecutive long edges enclosing at least 60, because that would imply $\alpha \leqslant \frac{360-120}{7} < 36$, a contradiction.

Therefore, at most one pair of consecutive long edges enclose an angle at least 60. We have 7 long edges, so there exists a long e_k such that e_{k-1} and e_{k+1} are long and $\angle e_{k-1}e_k = \angle e_ke_{k+1} = \alpha$. Then by Rule C, X_k can have at most 7 neighbors, at most 5 of which have degree at least 9.

It is also easy to see (by Rule D), that for each short edge e_m, X_m has a neighbor of degree at most 8. So the two vertices of degree 6, X_i and X_j contribute $2 \cdot \frac{-2}{5}$ to the Adjusted Degree of P, and X_k contributes $\frac{-1}{5}$, which add up to -1, so we are done.

(6) $\deg(P) = 9$ *and* $\deg_1(P) \leqslant 1$ *and* $\alpha > 36$.

We cannot have 2 pairs of consecutive edges enclosing angle 60, because $\alpha \leqslant \frac{360-120}{7} < 36$ is a contradiction. There are again two subcases.

(6.1) *There is a pair of long edges enclosing an angle at least* 60.

Now the remaining angles between consecutive edges must be either α or β.

If there are no short edges, then we have 7 edges so that the previous and successive edges are enclosing α with them. Thus by Rule C, the endpoints of these seven edges can have at most 7 neighbors each, so we are done.

If there is a short edge e_j, then by Rule B, X_j has at most 6 neighbors. Moreover, there are at least 4 long edges so that they enclose α with neighboring edges. These 4 long edges have at most degree 7 (by Rule C), and each has at most 6 neighbors with degree 9 (by Rule D). These 4 long edges and the short one contribute altogether (in the worst case) $4 \cdot \frac{-1}{6} - \frac{2}{6} = -1$ to the Adjusted Degree of P, so we are done.

(6.2) *There is no pair of long edges enclosing an angle at least* 60.

We have at least 8 long edges, so at least 6 of them enclose α with their neighbors. Therefore by Rules C and D the endpoints of these six edges are of degree at most 7, and each have at most 6 neighbors of degree at least 9. Thus the Adjusted Degree of P is at most $9 + 6 \cdot \frac{-1}{6} = 8$, so we are done. □

3. The golden ratio

Proof of Theorem 2. (i) First we prove the lower bound

$$4 \leqslant \sup_n \max_{|X''|=n} \frac{m_1 + m_2}{n}.$$

Observe that the configuration in Fig. 7 has 5 points with degree 9, and the rest of the points have degree 8. In Fig. 8,

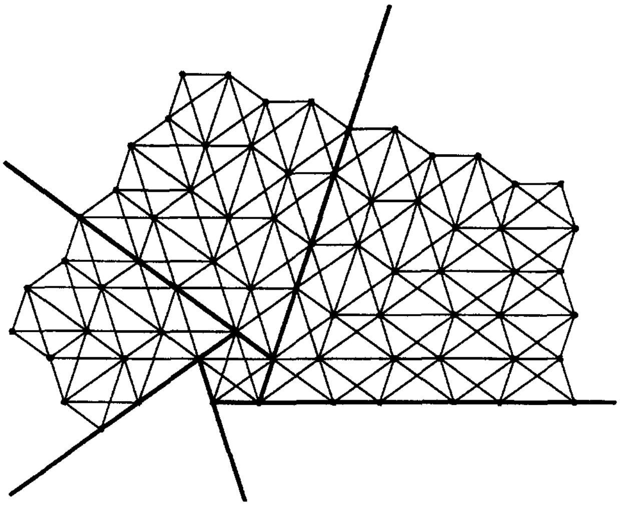

Fig. 7.

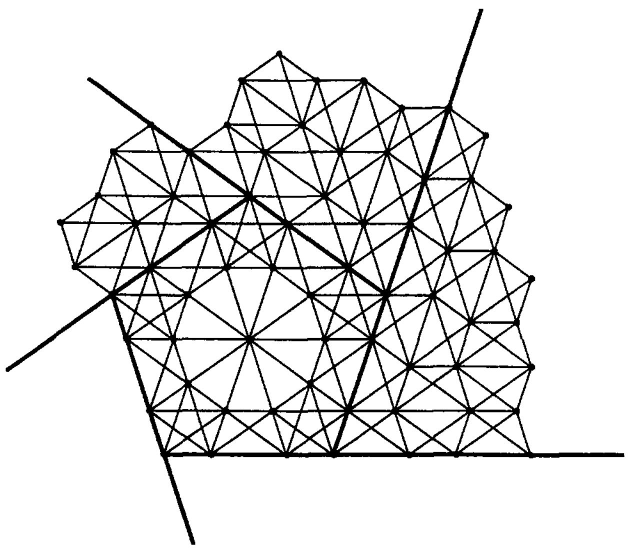

Fig. 8.

- 1 point has degree 10,
- 15 points have degree 9,
- 10 points have degree 7,

and all the other points have degree 8. This implies that

$$4 \leqslant \sup_n \max_{|X''|=n} [(m_1 + m_2)/n].$$

I conjecture, that this lower bound cannot be improved. Note, that for the above two configurations, although the average degree of the vertices does not exceed 8, the degrees of some points do, i.e., my conjecture is 'locally violated'.

(ii) Now we prove the upper bound

$$\sup_n \max_{|X''|=n} \frac{m_1+m_2}{n} \leqslant 4.5.$$

Let X be a set of n points where $d_1=1$ and $d_2=(\sqrt{5}+1)/2$. Let $G=G(X)$ be the graph whose vertices are the points of X, and two vertices are connected by an edge whenever their distance is either d_1 or d_2. We will use the notation of the proof of Theorem 1.

$d_2=(\sqrt{5}+1)/2$ implies that $\alpha=\beta=36$, $\delta=72$ and $\gamma=108$. Let P be an arbitrary vertex of G. Each pair of consecutive edges around P enclose either 36, or at least 60. This implies that $\deg(P)\leqslant 10$. If all vertices of G has degree at most 9, then we are done. Suppose that vertex P has degree 10.

Let $X_1,\dots,X_{10}$ denote the neighbors of P in clockwise order and let $e_i=PX_i$. Now $\angle X_iPX_{i+1}\geqslant 36$ implies that $\angle X_iPX_{i+1}=36$ for all i (i is taken modulo 10). Moreover, any two short edges incident to P enclose at least γ (since they cannot enclose 60), which implies that $\deg_1(P)\leqslant 3$.

We will prove that the Adjusted Degree of P (which is now $\deg(P)+\sum\{[\deg(X_i)-9]/t_i\}$) is at most 9. Then it follows that the average degree of vertices in G is at most 9, which implies the theorem.

Clearly, $\angle X_iPX_{i+1}\geqslant 36$ for all i, therefore $\frac{360}{10}=36$ implies that $\angle X_iPX_{i+1}=36$ for all i.

We introduce the following notation.

Let us call X_i (see Fig. 9)

- a *vertex of type A* if $|e_{i-2}|=|e_{i-1}|=|e_i|=|e_{i+1}|=|e_{i+2}|=d_2$;
- a *vertex of type B* if $|e_{i-3}|=|e_{i-2}|=|e_{i-1}|=|e_{i+1}|=|e_{i+2}|=|e_{i+3}|=d_2$ and $|e_i|=1$;
- a *vertex of type C* if $|e_{i-2}|=|e_i|=|e_{i+1}|=|e_{i+2}|=d_2$ and $|e_{i-1}|=1$;
- a *vertex of type D* if $|e_{i-1}|=|e_i|=|e_{i+1}|=|e_{i+2}|=d_2$ and $|e_{i-2}|=1$;
- a *vertex of type E* if $|e_i|=|e_{i+2}|=d_2$ and $|e_{i+1}|=1$;
- a *vertex of type F* if $|e_{i-2}|=|e_{i-1}|=|e_{i+1}|=|e_{i+2}|=|e_{i+3}|=d_2$ and $|e_i|=1$.

It is easy to see that the type A, type B and type C vertices are of degree at most 8 while the type D, type E and type F vertices are of degree at most 9.

Let $AA'B$ be a triangle with $|AB|=|A'B|=1$, $|AA'|=d_2$. Let C and C' be points that satisfy $|AC|=|A'C'|=d_2$ and $\angle CAB=\angle BA'C'=36$ (see Fig. 10). Then it follows that $|CC'|<1$. Therefore if A,B, and A' are elements of X, then at most one of C and C' may belong to X.

In such a case (when A, B and A' are elements of X), we will call the pair of vertices A and A' *twins*. Then either $\deg(A)\leqslant 9$ or $\deg(A')\leqslant 9$. Moreover, if C is a vertex of G, then the next neighbor of A' (following B in clockwise direction) can be

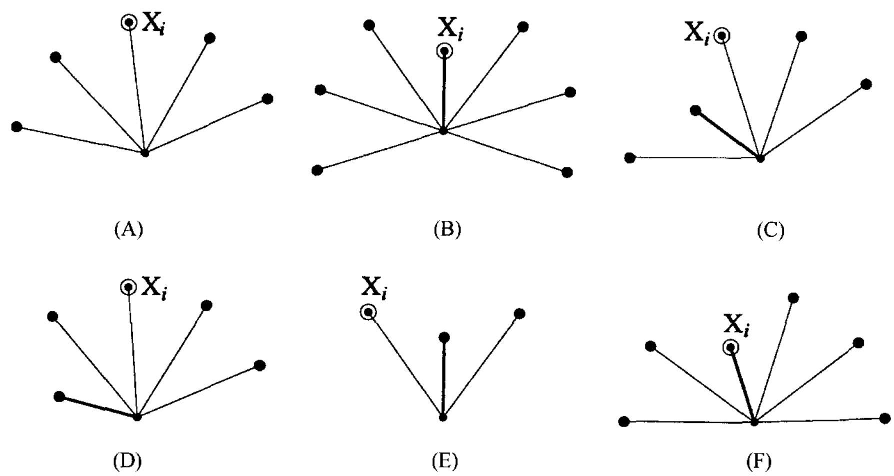

Fig. 9.

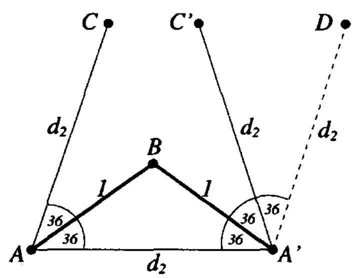

Fig. 10.

D, where $\angle DA'B = 72$, or a vertex even further from B. Intuitively this means that out of the 10 possible neighbors of A', one is missing.

We distinguish 4 cases.

(1) $\deg(P) = 10$ *and* $\deg_1(P) = 0$.

Now all vertices $X_1, \ldots, X_{10}$ are of type A, therefore the Adjusted Degree of P is at most $\deg(P) + 10 \cdot \frac{-1}{8} < 9$, so we are done.

(2) $\deg(P) = 10$ *and* $\deg_1(P) = 1$.

We may suppose that $|e_1| = 1$. Then vertices $X_4, \ldots, X_8$ are of type A, X_1 is of type B, while X_2 and X_{10} are type C vertices. There are 8 vertices of degree at most 8, so they contribute at most $8 \cdot \frac{-1}{8}$ to the Adjusted Degree of P.

(3) $\deg(P) = 10$ *and* $\deg_1(P) = 2$.

We may suppose that $|e_1| = 1$ and that the other short edge is one of e_4, e_5 or e_6. So there are 3 subcases.

(3.1) $|e_6| = 1$.

Now X_1 and X_6 are of type B, X_2, X_5, X_7 and X_{10} are of type C, so these 6 vertices have degree at most 8. The remaining vertices are of type D, so they are of degree

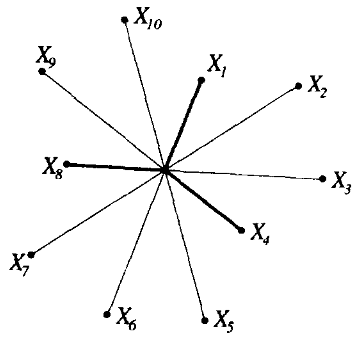

Fig. 11.

at most 9. Therefore all 6 vertices of degree at most 8 have at least 2 neighbors with degree at most 9, thus the Adjusted Degree of P is at most $10+6\cdot\frac{-1}{6}=9$.

(3.2) $|e_5|=1$.

Now X_1 and X_5 are of type B, X_8 is type A, X_2,X_4,X_6 and X_{10} are of type C. So we have 7 vertices with degree at most 8, and all have at most 7 neighbors with degree at least 10. Therefore the Adjusted Degree of P is at most 9.

(3.3) $|e_4|=1$.

Now vertices X_7 and X_8 are of type A, X_5 and X_{10} are type C, therefore they have degree at most 8. X_2 and X_3 are type E, X_1 and X_4 are type F, X_6 and X_9 are of type D, therefore they have degree at most 9. This implies that $\sum \deg(X_i)\leqslant 4\cdot 8+6\cdot 9$, but we have 4 pairs of twin vertices: (X_1,X_9), (X_1,X_3), (X_4,X_2), (X_4,X_6), which means an other improvement of 4, so $\sum \deg(X_i)\leqslant 4\cdot 8+6\cdot 9-4$. This implies $\deg(P)+\sum_{i=1}^{10}\{[\deg(X_i)-9]/t_i\}\leqslant 9$, so we are done.

(4) $\deg(P)=10$ *and* $\deg_1(P)\geqslant 3$.

It is easy to see, that the only possibility (up to symmetry) is that e_1, e_4 and e_8 are short edges, and the others are long (see Fig. 11). Then X_5 and X_7 are type C vertices of degree at most 8. Vertices X_4 and X_8 are of type F, X_2,X_3,X_9 and X_{10} are type E vertices, so these 6 vertices have degree at most 9. X_1 and X_6 have degree at most 10. On the other hand, we have 6 pairs of twin vertices (X_1,X_3), (X_1,X_9), (X_4,X_2), (X_4,X_6), (X_8,X_6), (X_8,X_{10}), which means that one edge is missing for each twin pair. So $\sum \deg(X_i)$ is at least 6 less than the previously obtained bound, therefore $\sum \deg(X_i)\leqslant(2\cdot 8+2\cdot 10+6\cdot 9)-6$.

It is also easy to see that every vertex X_i has at least 2 neighbors among the other $\{X_i\}$ with degree at most 9. Then

$$\deg(P)+\sum\frac{\deg(X_i)-9}{t_i}\leqslant 9,$$

where the sum is taken over those X_i for which $\deg(X_i)\leqslant 8$.

Thus the theorem is proved. □

4. The case d_2 is large

Proof of Theorem 3. It easily follows from the configuration on Fig. 2. □

Proof of Theorem 4. Let X be a set of n points where $d_1 = 1$ and $d_2 > \sqrt{3}$. As before, let $G = G(X)$ be the graph whose vertices are the points of X, and two vertices are connected by an edge whenever their distance is either d_1 or d_2. We will use the notation of the proof of Theorem 1.

Let P be an arbitrary vertex of G. Let $X_1, X_2, \ldots$ denote the neighbors of P in clockwise order and let $e_i = PX_i$.

We distinguish 2 cases.

Case 1. $d_2 > 1/(2\sin 15) = 1.93185$.

In this case $\alpha < 30$, $\beta < 15$, $\gamma > 150$ and $\delta > 75$. Note that if $\alpha < 28$, then $d_2 > 2$, and therefore no triangle exists with one side of length d_2 and two sides of length 1. Clearly, two short edges incident to P enclose either 60, or at least γ, so $\deg_1(P) \leqslant 2$.

Observe that two long edges enclose either α, or at least 60. Suppose that e_j, e_{j+1} and e_{j+2} are long edges. If $\angle e_j e_{j+1} = \angle e_{j+1} e_{j+2} = \alpha$, then $\angle e_j e_{j+2} = 2\alpha < 60$, which is impossible, so we conclude that $\angle e_j e_{j+2} \geqslant \alpha + 60$.

A long and a short edge enclose either β, or at least δ. Suppose e_i is a short edge, but $e_{i+1}, e_{i+2}, e_{i+3}$ and e_{i+4} are long edges (see Fig. 12a). Then either $\angle e_i e_{i+1} = \beta$ or $\angle e_i e_{i+1} \geqslant \delta$, but in both cases $\angle e_i e_{i+2} \geqslant \delta$. Similarly as before, $\angle e_{i+2} e_{i+4} \geqslant \alpha + 60$, therefore

$$\angle e_i e_{i+4} \geqslant \delta + \alpha + 60. \tag{a}$$

Moreover, if $\alpha < 28$, then $\angle e_i e_{i+1} \neq \beta$, so $\angle e_i e_{i+1} > \beta$, and we get the inequality

$$\angle e_i e_{i+3} \geqslant \delta + \alpha + 60. \tag{b}$$

Similarly, if we suppose that e_i is short, but $e_{i-2}, e_{i-1}, e_{i+1}$ and e_{i+2} are long (see Fig. 12b), then we get that $\angle e_i e_{i+2} \geqslant \delta$. At most one of e_{i+1} and e_{i-1} can enclose an angle β with e_i, so we may suppose that $\angle e_{i-1} e_i \geqslant \delta$. Then it is easy to see that $\angle e_{i-2} e_i \geqslant \delta + \alpha$, therefore

$$\angle e_{i-2} e_{i+2} \geqslant 2\delta + \alpha = 180. \tag{c}$$

We distinguish 3 subcases.

(1.1) $\deg_1(P) = 2$.

Let e_i and e_j be the short edges.

- If $\angle e_i e_j = 60$, then it follows that $j = i + 1$. Now by (a) we get $\angle e_{i+1} e_{i+5} \geqslant \delta + \alpha + 60$ and $\angle e_{i-4} e_i \geqslant \delta + \alpha + 60$. Supposing that $e_{i-4}, e_{i-3}, \ldots, e_{i+4}$ are all different edges (i.e. $\deg(P) > 9$), we would get

  $$360 \geqslant 60 + 2(\delta + \alpha + 60) = 180 + 2\delta + 2\alpha = 360 + \alpha,$$

  a contradiction. Therefore in this case $\deg(P) \leqslant 8$.

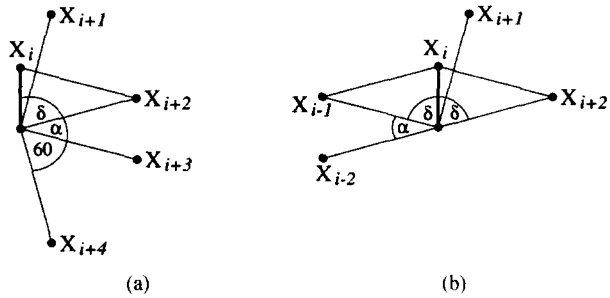

Fig. 12.

• If $\angle e_ie_j > 60$, then $\angle e_ie_j > 150$, and by using (c), some straigthforward calculation gives that again $\deg(P) \leqslant 8$.

(1.2) $\deg_1(P) = 1$.

Suppose e_i is short. Since $\angle e_ie_{i+1} = \angle e_{i-1}e_i = \beta$ is impossible, without loss of generality we may suppose that $\angle e_{i-1}e_i > \beta$. Then equations (a) and (b) imply $\angle e_ie_{i+4} \geqslant \delta + \alpha + 60$ and $\angle e_{i-3}e_i \geqslant \delta + \alpha + 60$.

Supposing that $e_{i-3}, e_{i-2}, \ldots, e_{i+4}$ are all different edges (therefore $\deg(P) \geqslant 8$), we get that

$$\angle e_{i+4}e_{i-3} \leqslant 360 - 2(\delta + \alpha + 60) = 240 - 2\delta - 2\alpha = 60 - \alpha.$$

Thus there cannot be more edges between e_{i+4} and e_{i-3}, so $\deg(P) \leqslant 8$.

(1.3) $\deg_1(P) = 0$.

In this case $\angle e_ie_{i+2} \geqslant 60 + \alpha$ for every i, therefore $\deg(P) \leqslant 10$. We distinguish two subcases.

(1.3.a) $\deg(P) = 10$.

Easy to see that the only possible arrangement for this is the one shown on Fig. 13a (where 60+ denotes an angle at least 60). This implies that $\alpha < 12$, so $d_2 > 2$.

Vertex X_1 has a short edge attached to it. By (1.1) and (1.2), and using the fact $d_2 > 2$, this implies $\deg(X_1) \leqslant 7$. However, here the next neighbor of X_1, following P in counterclockwise direction is Y (possibly $Y = X_2$), where $\angle YX_1P \geqslant 60$. Some straightforward calculation (similar to the one used in parts (1.1) and (1.2)) gives that in this case $\deg(X_1) \leqslant 6$.

This is true for all X_i, so P has 10 neighbors of degree at most 6, thus the Adjusted Degree of P is at most $10 + 10 \cdot \frac{-2}{6} \leqslant 8$.

(1.3.b) $\deg(P) = 9$.

Now the only possible arangement is shown on Fig. 13b. Again $d_2 > 2$, and by a similar argument as for part (i) we get that the Adjusted Degree of P is at most $9 + 4 \cdot \frac{-2}{6} < 8$.

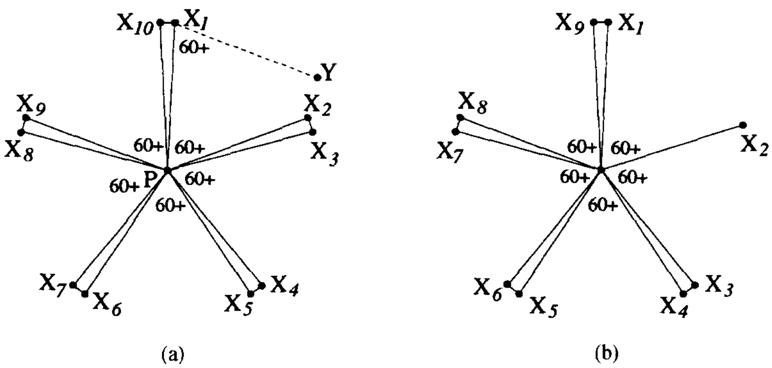

Fig. 13.

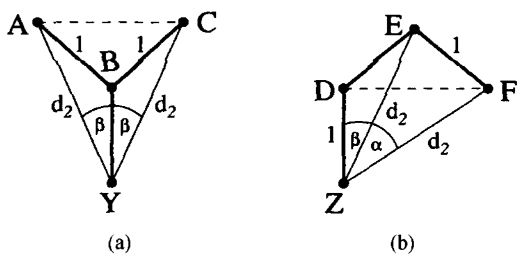

Fig. 14.

Case 2. $1/(2\sin 15) > d_2 > \sqrt{3}$.

In this case $30 < \alpha < 33.56$, $15 < \beta < 30$, $150 > \gamma > 120$ and $75 > \delta > 73$. Straightforward calculations give the following useful inequalities:

$$\alpha + \beta < \delta, \quad 60 > 2\beta > \alpha, \quad \delta + \beta > 90.$$

Let us make two observations first.

Let Y, A, B and C be four points such that $|YA| = |YC| = d_2$, $|YB| = 1$ and $\angle AYB = \angle BYC = \beta$ (Fig. 14a). In this case $1 < |AC| < d_2$, so Y, A, B and C cannot all be elements of the set X.

Let Z, D, E and F be four points such that $|ZE| = |ZF| = d_2$, $|ZD| = 1$, $\angle DZE = \beta$ and $\angle EZF = \alpha$ (see Fig. 14b). In this case $1 < |DF| < d_2$, so Z, D, E and F cannot all be elements of the set X.

In the sequel we will often use the fact that Figs. 14a and 14b are 'forbidden' configurations.

Consider the neighbors of vertex P. Call X_i a *vertex of type G*, if e_{i-2}, e_{i-1}, e_i and e_{i+1} are long edges and $\angle e_{i-2}e_{i-1} = \angle e_{i-1}e_i = \angle e_ie_{i+1} = \alpha$ (see Fig. 15).

Lemma. *Any vertex of type G has at most 6 neighbors.*

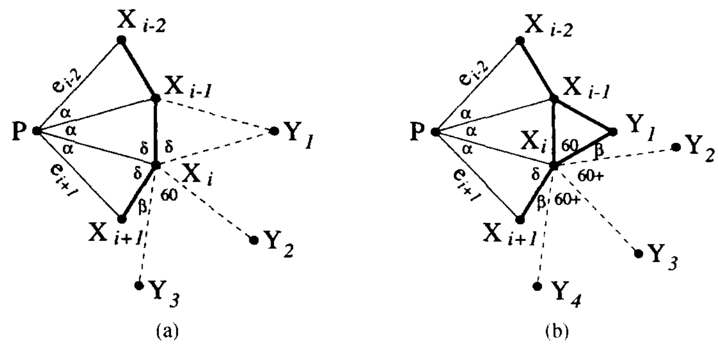

Fig. 15.

Proof of the Lemma. Suppose X_i is a vertex of type G. We may suppose that $\deg(X_i) \geqslant 6$. There are two cases.

(a) Suppose first that only two short edges are adjacent to X_i (namely X_iX_{i-1} and X_iX_{i+1}), as shown on Fig. 15a. In this case let $Y_2, Y_3, X_{i+1}, P, X_{i-1}$ and Y_1 denote consecutive neighbors of X_i in clockwise order. Easy to see that $\angle X_{i-1}X_iY_1 \geqslant \delta$, $\angle Y_3X_iX_{i+1} \geqslant \beta$ and so $\angle Y_2X_iX_{i+1} \geqslant \beta + 60$. Hence,

$$\angle Y_1X_iY_2 \leqslant 360 - (3\delta + \beta + 60) < 210 - 2\delta \leqslant 30 + \alpha < 2\alpha,$$

and there is no space for one more edge between X_iY_1 and X_iY_2, so $\deg(X_i) \leqslant 6$.

(b) Suppose that there is at least one more short edge adjacent to X_i. The only possibility is that it encloses 60 with either X_iX_{i-1} or X_iX_{i+1}. It is easy to see that there is no space for additional short edges. We may suppose that the short edge encloses 60 with X_iX_{i-1} (the other case can be treated similarly) (see Fig. 15b).

To get a contradiction, suppose $\deg(X_i) \geqslant 7$. Let $Y_3, Y_4, X_{i+1}, P, X_{i-1}, Y_1$ and Y_2 be consecutive neighbors of X_i. Then clearly $\angle Y_1X_iY_2 \geqslant \beta$ and as before, $\angle Y_3X_iX_{i+1} \geqslant \beta + 60$. This gives that

$$\angle Y_2X_iY_3 \leqslant 360 - (2\delta + 2\beta + 120) < 60,$$

a contradiction. Thus $\deg(X_i) \leqslant 6$. $\square$

Now returning to the proof of the theorem, suppose, in order to get a contradiction, that $\deg(P) = k \geqslant 9$. We distinguish 7 cases, according to the value of $\deg_1(P)$.

(2.1) $\deg_1(P) = 0$.

Since $\alpha > 30$, $\deg(P) < 12$. There are 3 subcases.

(2.1.a) $\deg(P) = 11$.

It is possible only if $\angle X_iPX_{i+1} = \alpha$ for all i. Then all the X_i are vertices of type G, so the Lemma gives that $\deg(X_i) \leqslant 6$ for all i, therefore the Adjusted Degree of P is at most $11 + 11 \cdot \frac{-2}{6} < 8$.

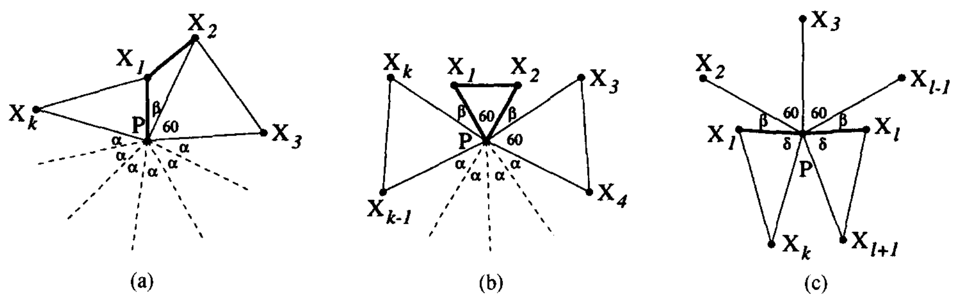

Fig. 16.

(2.1.b) $\deg(P)=10$.
There is at most one j for which $\angle X_jPX_{j+1}>\alpha$ (in fact, $\angle X_jPX_{j+1}\geqslant 60$). It is easy to see that there are still at least 8 vertices of type G, so the Adjusted Degree of P is at most $10+8\cdot\frac{-2}{6}<8$.
(2.1.c) $\deg(P)=9$.
Now at most 2 of the angles $\angle X_iPX_{i+1}$ are larger than α (so at least 60). It is easy to see that even in this case we still have at least 3 vertices of type G. Thus, the Adjusted Degree of P is at most $9+3\cdot\frac{-2}{6}=8$.

(2.2) $\deg_1(P)=1$.
We may suppose that e_1 is short. The two forbidden configurations of Fig. 14 imply that either $\angle e_ke_1\geqslant\delta$ or $\angle e_1e_2\geqslant\delta$. We may suppose the first. Then clearly $\angle e_1e_3\geqslant\beta+60$, as it is shown on Fig. 16a. This gives

$$\angle X_3PX_k\leqslant 360-(\delta+\beta+60)<210.$$

Each pair of consecutive edges between e_3 and e_k enclose at least α, therefore $\deg(P)\leqslant 9$. Supposing $\deg(P)=9$, we get that $\angle X_iPX_{i+1}=\alpha$ for each i, where $3\leqslant i\leqslant k-1$ and then X_i $(4\leqslant i\leqslant k-1)$ is a type G vertex, so the Adjusted Degree of P is at most $9+5\cdot\frac{-2}{6}<8$.

(2.3) $\deg_1(P)=2$.
There are two subcases.
(2.3.a) *The two short edges enclose 60 (see Fig.* 16b).
Since the configuration on Fig. 14b. is forbidden, $\angle X_2PX_4\geqslant\beta+60$ and $\angle X_{k-1}PX_1\geqslant\beta+60$, so

$$\angle X_4PX_{k-1}\leqslant 360-(2\beta+180)<150.$$

Thus, there are at most 3 edges between X_4 and X_{k-1}, therefore $k\leqslant 9$. Moreover $k=9$ is possible only when $\angle X_iPX_{i+1}=\alpha$ for all $4\leqslant i\leqslant k-2$. This implies that the vertices X_i $(5\leqslant i\leqslant 7)$ are of type G, so the Adjusted Degree of P is at most $9+3\cdot\frac{-2}{6}=8$.

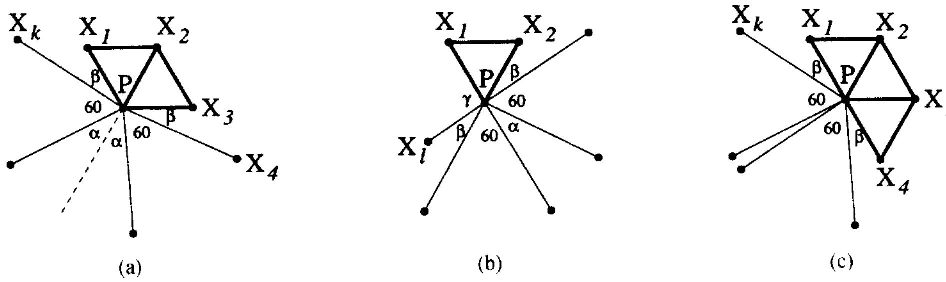

Fig. 17.

(2.3.b) *The two short edges (say e_1 and e_l) enclose more than* 60 (*see Fig.* 16c).

Clearly $\angle X_1PX_l \geqslant \gamma$ and $\angle X_lPX_1 \geqslant \gamma$. Then $l \geqslant 7$ would imply $\angle X_1PX_3 \geqslant 60+\beta$ and $\angle X_{l-2}PX_l \geqslant 60+\beta$, and naturally $\angle X_3PX_{l-2} \geqslant 2\alpha$ (since $l-2 \geqslant 5$). Therefore

$$\angle X_lPX_1 \leqslant 360 - 2(60+\beta) - 2\alpha < 180 - 2\beta = \gamma,$$

a contradiction. Thus, $l \leqslant 6$, and by the symmetry of the configuration we may suppose $l=6$.

Now either $\angle X_{k-1}PX_1 \geqslant \delta+\alpha$ or $\angle X_1PX_3 \geqslant \delta+\alpha$. In any case $\angle X_{k-1}PX_3 \geqslant (\delta+\alpha)+(\beta+60) > 180$. We get similarly $\angle X_4PX_8 > 180$. But then $\angle X_3PX_4 < 0$, a contradiction.

(2.4) $\deg_1(P)=3$.

There are 3 subcases.

(2.4.a) *There are* 3 *consecutive short edges, e_1, e_2 and e_3* (*Fig.* 17a).

In this case $\angle X_{k-1}PX_1 \geqslant \beta+60$ and $\angle X_3PX_5 \geqslant \beta+60$, so

$$\angle X_5PX_{k-1} \leqslant 360 - 120 - 2(\beta+60) < 90.$$

However, $k-1 \geqslant 8$ implies $\angle X_5PX_{k-1} \geqslant 3\alpha > 90$, a contradiction.

(2.4.b) *There are only two consecutive short edges.*

Suppose e_1, e_2 and e_l are short, where $3<l<k$ (Fig. 17b). By symmetry, we may suppose $l \geqslant 6$. $l \geqslant 7$ is impossible, because $\angle X_lPX_1 \geqslant \gamma$, $\angle X_2PX_4 \geqslant \beta+60$, $\angle X_{l-2}PX_l \geqslant \beta+60$ and $\angle X_4PX_{l-2} \geqslant \alpha$, which gives

$$360 \geqslant 60+\gamma+2(\beta+60)+\alpha = (\gamma+2\beta)+180+\alpha = 360+\alpha,$$

a contradiction.

So the only possibility is $l=6$, but then still $\angle X_2PX_4 \geqslant \beta+60$, $\angle X_4PX_l \geqslant \beta+60$, $\angle X_lPX_{l+2} \geqslant \beta+60$ and $\angle X_{k-1}PX_1 \geqslant \beta+60$. Thus,

$$360 \geqslant 4(\beta+60)+60 = 300+4\beta > 360,$$

a contradiction.

(2.4.c) *No two of the short edges enclose* 60.

Now if e_1, e_l and e_t are the short edges, then $\angle e_1 e_l \geqslant \gamma > 120$, $\angle e_l e_t \geqslant \gamma > 120$ and $\angle e_t e_1 \geqslant \gamma > 120$, a contradiction.

(2.5) $\deg_1(P) = 4$.

It is easy to see that the only possible arrangement is when the short edges are consecutive: say, e_1, e_2, e_3 and e_4 (see Fig. 17c).

We get that $\angle X_1PX_4 = 180$, $\angle X_4PX_6 \geqslant \beta + 60$, $\angle X_{k-1}PX_1 \geqslant \beta + 60$ and $\angle X_6PX_{k-1} \geqslant \alpha$. Then it follows that

$$360 \geqslant 180 + 2(\beta + 60) + \alpha = 300 + 2\beta + \alpha > 360,$$

a contradiction.

(2.6) $\deg_1(P) = 5$.

In any arrangement of exactly 5 short edges, there are two enclosing an angle more than 60, but less than γ, a contradiction.

So a configuration with $\deg_2(P) = 5$ does not exist.

(2.7) $\deg_1(P) = 6$.

In this case there is no space for long edges, so $\deg(P) = 6$.

Thus, the proof of Theorem 4 is complete. □

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